Area scaling entropies for gravitating systems

Jonathan Oppenheim

Theoretical Physics Institute, University of Alberta, 412 Avadh Bhatia Physics Lab.,

Alta., Canada, T6G 2J1. Email: jono@phys.ualberta.ca

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Abstract

The entropy of a spherically symmetric distribution of matter in self-equilibrium is calculated. When gravitational effects are neglected, the entropy of the system is proportional to its volume. As effects due to gravitational self-interactions become more important, the entropy acquires a correction term and is no longer purely volume scaling. In the limit that the boundary of the system approaches its event horizon, the total entropy of the system is proportional to its area. The scaling laws of the system's thermodynamical quantities are identical to those of a black hole, even though the system does not possess an event horizon.

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The entropy of ordinary matter is usually proportional to its volume and depends crucially on its composition. The entropy of a black hole [1] [2] is therefore very mysterious, in part because it is proportional to the black hole's area A. In order to explain this peculiar behavior, researchers have invoked holography, string theory, entanglement entropy [3], and brick-wall models [4].

However, the volume scaling properties for the entropy of ordinary matter depend on the fact that interactions are short range in comparison to the size of the system, or that the interactions are screened. For long-range interactions such as electromagnetism, the entropy still scales as the system's volume as long as the net charge of the system is neutral [5], but little is known about the thermodynamic properties in other cases where long range interactions are present.

In this paper, we will examine a system where long range gravitational interactions are present. Although black hole entropy appears to possess very peculiar properties, we will see that many of these features are present in ordinary self-gravitating systems which do not possess an event horizon. In particular, we will study spherically symmetric matter distributions which are in self-equilibrium. In the absence of gravity, the entropy of the distribution is assumed to be proportional to its volume. However, when the system's own weak gravitational field is taken into account, the entropy acquires a correction term, spoiling its volume-scaling properties.

In the strong field limit, the results are even more surprising. In the limit that the system is about to form a black hole, its entropy is proportional to its area for a large class of equations of state. The scaling laws of the system's temperature and energy are also identical to those of a black hole despite the fact that no horizon is present. This suggests that many of the peculiar properties of black hole thermodynamics are the result of gravitational self-interactions, rather than the horizon.

In this work, the system is modeled as a dense series of thin shells of arbitrary composition. This allows the system to be compressed to a size close to its own event horizon [6]. The work in this paper is related to reference [7] where a single shell is considered. In that

case, the entropy is assumed to always be proportional to the system's area, and when the shell is brought close to its event horizon, the constant of proportionality is found to be 1/4.

We consider n densely packed shells, where the radius of the i'th shell is given by r_i, and its thermodynamical quantities, as measured by local observers on each shell are given by E_i (mass-energy), T_i (temperature), S_i (entropy), P_i (tangential pressure), N_i (particle number), and μ_i (chemical potential). The entire configuration is assumed to be in equilibrium with itself, and therefore, it supports itself in its own gravitational field. If the material has a rest mass, then this corresponds to a constant term in the chemical potential. One could also consider an arbitrary number of particle species, however the effect of this on our calculation is trivial.

We assume that in the absence of gravity, the system has no unscreened long-range interactions and is therefore extensive. In other words, the entropy, particle number and energy scale as the size of the system when the intensive variables (temperature, pressure and chemical potential) are held fixed. In most applications of thermodynamics this is almost always assumed to be the case (at least implicitly), for without it, taking the thermodynamic limit becomes problematic.

We will first calculate the local thermodynamical quantities in terms of observables at infinity. Spherical symmetry dictates that the metric outside the i'th shell is

$$\begin{aligned} ds^2 &= {}_i g_{\mu\nu} dx^\mu dx^\nu \\ &= -c_i k_i^2(r) dt^2 + k_i^{-2}(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{1}$$

where throughout this paper we use units $G = c = \hbar = k_B = 1$. In order for the time coordinate t to be continuous throughout the space time, the c_i need to satisfy the condition

$$c_i k_i^2(r_{i+1}) = c_{i+1} k_{i+1}^2(r_{i+1}) \tag{2}$$

(an overall constant is not physically relevant, and just scales the time coordinate). If ${}_iT_{ab}$ is the intrinsic stress-energy tensor on the i'th shell, then its components are related to the junction conditions of the extrinsic curvature and intrinsic metric [8]

$$8\pi{}_iT_{ab} = {}_i[g_{ab}K - K_{ab}] \tag{3}$$

where $a = 0..2$ and the notation ${}_i[\ ]$ indicates the jump in the bracketed quantity evaluated at r_i e.g. ${}_i[k(r)] \equiv {}_ik(r_i) - {}_{i-1}k(r_i)$. Since the shells are static, the local energy E_i of the i'th shell is $-4\pi r_i^2 {}_iT^t_t$, while the tangential pressure P_i is ${}_iT^\phi_\phi = {}_iT^\theta_\theta$. The jump condition thus give

$$E_i = -{}_i[k(r)]r_i \tag{4}$$

$$8\pi P_i = {}_i\left[k'(r) + \frac{k(r)}{r}\right] \ . \tag{5}$$

If we take the material to have zero net charge (as indeed we must, for the system to be locally extensive), and have no angular momentum, then the $k_i(r)$ are of the Schwarzschild form

$$k_i(r) = \sqrt{1 - \frac{2m_i}{r}} \ . \tag{6}$$

Using (6) we can then find ${}_ig_{tt}$ by calculating the constants given in (2)

$$c_i = \frac{(r_{i+1} - 2m_{i+1})(r_{i+2} - 2m_{i+2})\cdots(r_n - 2m_n)}{(r_{i+1} - 2m_i)(r_{i+2} - 2m_{i+1})\cdots(r_n - 2m_{n-1})} \tag{7}$$

(with $c_n = 1$).

The energy, and pressure of each shell are then

$$E_i = \sqrt{r_i}\left(\sqrt{r_i - 2m_{i-1}} - \sqrt{r_i - 2m_i}\right) \tag{8}$$

$$P_i = \frac{m_{i-1} - r_i}{8\pi r_i^2 k_{i-1}(r_i)} - \frac{m_i - r_i}{8\pi r_i^2 k_i(r_i)} \ . \tag{9}$$

The latter equation can be thought of as the condition for mechanical equilibrium, and ensures that the pressure balances the effect of gravity so that the configuration is at least momentarily static. The conditions for thermodynamic and chemical equilibrium are the Tolman relation [9]

$$T_i = \frac{T_\infty}{\sqrt{-_i g_{tt}(r_i)}} \quad . \tag{10}$$

and the equality of the red-shifted chemical potentials

$$\mu_i \sqrt{_i g_{tt}} = \mu_j \sqrt{_j g_{tt}} \quad . \tag{11}$$

The latter ensures that there will be no net particle flow (one can imagine that particles can hop from shell to shell via wires, or simply by jumping). The Tolman relation fixes the temperature of the shells with respect to the temperature at infinity T_∞ which we assume to be finite (to be rigorous, one could imagine a reservoir at infinity which is attached via a heat-conducting wire to all the shells).

These thermodynamical variables can be related to each other by the Gibbs-Duhem relation [10]

$$E_i = T_i S_i - A_i P_i + \mu_i N_i \tag{12}$$

where A_i is the area of each shell. This equation holds for systems which are locally extensive. To derive this, we hold the intensive variables (P_i, T_i, μ_i) fixed, and note that if we scale the size of a shell by λ, the entropy and energy will scale by the same amount. I.e.

$$E_i(\lambda S_i, \lambda A_i) = \lambda E_i(S_i, A_i) \quad . \tag{13}$$

We can then differentiate the above expression by λ to obtain the Gibbs-Dehum relation (which is an Euler relation of homogeneity one). In the presence of gravity, the equivalence principle ensures that the Gibbs-Duhem relation is valid as it applies to thermodynamical variables as measured in the proper rest frame of the shell. These variables are scalars, and are therefore frame-independent.

Substituting the expressions for the local variables into (12) one can write the entropy of each shell as

$$S_i = \frac{1}{2T_\infty}\left[(3m_i - r_i) - (3m_{i-1} - r_i)\frac{k_i(r_i)}{k_{i-1}(r_i)}\right]\sqrt{c}_i - \frac{\mu_n N_i}{T_\infty}\sqrt{1 - \frac{2m_n}{r_n}} \tag{14}$$

and the total entropy of the system is just the sum

$$S = \sum_i S_i \quad . \tag{15}$$

When gravitational effects are negligible, i.e. the configuration of shells is large in comparison to its Schwarzschild radius, S_i can be expanded to first order in m_i/r_i. It is instructive to look at some of the terms in equation (12) individually. The local energy can be expanded to give

$$E_i \simeq m_i - m_{i-1} + \frac{m_i^2 - m_{i-1}^2}{2r_i}. \tag{16}$$

$m_i - m_{i-1}$ can be interpreted as the additional energy (as measured at infinity) required to add the i'th shell to the configuration. Since to first order, m_i is just the sum of all E_j, $j \leq i$, we see that we have recovered the usual Newtonian expression

$$m_i - m_{i-1} = E_i + \phi_i \tag{17}$$

where ϕ_i is the Newtonian gravitational potential

$$\phi_i = -E_i \frac{E_1 + E_2 + ...E_{i-1}}{r_i} - \frac{E_i^2}{2r_i} \quad . \tag{18}$$

To first order in m_i/r_i, the pressure term also gives the usual Newtonian expression

$$P_i = -\frac{\phi_i}{2A_i} \tag{19}$$

The chemical potential term

$$\mu_n \sqrt{1 - \frac{2m_n}{r_n}} \simeq \mu_n (1 - \frac{m_n}{r_n}) \tag{20}$$

contains a correction which is equivalent to the Newtonian chemical potential due to an external gravitational field, although in this case, the gravitational field couples not only to the mass of each particle, but to the entire chemical potential μ_n.

One can also expand the expression for the temperature, and after substituting these terms in (14) and taking the sum over i, we arrive at the total entropy. To zeroth order

$$S \simeq T_\infty^{-1}(m_n - \mu_n N) \tag{21}$$

where N is the total number of particles. We see that when we ignore the first order effects due to gravity, the entropy, when expressed in terms of observables at infinity, is given as an Euler relation of homogeneity one. This demonstrates that the system is purely extensive. When the intensive variables (T_∞ and μ_n) are held fixed, the entropy is proportional to the total energy and total number of particles. If all the shells have the same energy density and number density, the entropy would scale as the volume of the system, where the volume is understood to be the sum of all the A_i.

However, if we include the first order terms, we find

$$S \simeq T_\infty^{-1}(m_n - \mu_n N + \frac{\phi}{2} - \mu_n N \frac{m_n}{r_n}) \tag{22}$$

where ϕ is the total gravitational energy required to construct the system

$$\phi = \sum_i \phi_i \quad . \tag{23}$$

The entropy includes correction terms, due to the gravitational self-interactions, and is no longer purely extensive.

We now go to the extreme limit when the outer shell approaches its own Schwarzschild radius ($r_n \to 2m_n$). For the purposes of this paper, we will assume that our system can approach its Schwarzschild radius, although more physically reasonable models may not be able to generate such intense pressure or exist at such high temperatures. Inspecting equation (7) we see that all the c_i except c_n are zero because of the $r_N - 2m_N$ term. As a result, from equation (14) we see that all the S_i except S_n are zero, although for now, we will retain the chemical potential terms. Substituting this result in (15) we can easily sum over i and calculate the total entropy

$$S = \frac{m_n}{2T_\infty} - \frac{\mu_n N}{T_\infty}\sqrt{1 - \frac{2m_n}{r_n}} \quad . \tag{24}$$

Pretorius, Vollick and Israel consider specific and general equations of state where the chemical potential term in equation (24) vanishes entirely [7], However, we can also demand the less restrictive condition that the chemical potential not be overly divergent i.e.

$$\lim_{r_n \to 2m_n} \mu_n N \sqrt{1 - \frac{2m_n}{r_n}} \ll m_n \quad . \tag{25}$$

As a result, the only term which contributes to equation (15) is the entropy of the final shell S_n:

$$S = \frac{m_n}{2T_\infty} \quad . \tag{26}$$

Since the total entropy is equal to the entropy of the final shell, we see that the entropy is proportional to the area of the system. That this is the case, can also be seen by noting that equation (26) is an Euler relation of homogeneity two. This is the precise relation which governs the entropy and temperature of black holes. Applying the first law $dm_n = T_\infty dS$ to (26), one finds that (in terms of a constant γ)

$$\begin{aligned} S &= \gamma m_n^2 \\ &= \frac{\gamma}{16\pi} A_n \end{aligned} \tag{27}$$

and

$$T_\infty = (2\gamma m_n)^{-1} \tag{28}$$

where as before, m_n is the total ADM mass as measured at infinity, and A_n is the area of the final shell (and therefore, of the entire system).

These relations, up to the constant γ, are identical to the ones which govern Schwarzschild black holes in $3 + 1$ dimensions. Here however, the entropy is not a property of a horizon, but is the logarithm of the number of micro-states of our system. The temperature, which is usually considered to be independent of the size of the system, is now inversely proportional to the mass of the system. This result is even more remarkable, since it has not been derived by considering quantum fields on the space-time. These later two equations are the central result of this paper.

It is also interesting to note that if, at the instant before the system forms a black hole, it has the Hartle-Hawking temperature

$$T_{HH} = 2m_n/A_n \tag{29}$$

then equations (28) and (27) imply that its entropy before black hole formation is

$$S(T_{HH}, m_n) = A_n/4 \quad . \tag{30}$$

This is identical to the entropy of a black hole of equivalent mass.

Indeed, Pretorius, Vollick, and Israel argue that equation (29) is the appropriate temperature for a shell as $r_n \to 2m_n$. They consider quantum fields propagating on the space-time, and demand that that the shell be in equilibrium with the local acceleration temperature [11].

Ordinarily, the appropriate vacuum state for a spherically symmetric distribution without a horizon is the Boulware vacuum (which has a temperature of zero at infinity). However, when the boundary of our system lies close to its Schwarzschild radius, one cannot consistently construct the Boulware vacuum, since the stress-energy of the quantum fields will be infinite (and negative), and will depend on the number of fields. These trans-plankian frequencies will result in an enormous and incalculable back-reaction. It is therefore unclear what the appropriate vacuum state and vacuum temperature are. Pretorius, Vollick, and Israel demand that the quantum field be in the Hartle-Hawking state with a temperature at infinity given by equation (29), however, this requires adding layers of insulation to the shells, and positive energy to the field to compensate for the back-reaction.

The manner in which this is done, and the resulting temperature of the shells is not unique. For example, for a single shell just outside its Schwarzschild horizon, consistently constructing the Hartle-Hawking state requires adding a thin layer of insulation to the shell. The insulation prevents the fields inside the shell from having a diverging temperature, which would result in an infinite back-reaction. If the insulation is added to the outside of the shell, the temperature of the shell can be zero, since it is shielded from the hot field just outside the shell. If the insulation is added to the inside of the shell, the local shell temperature will match the local temperature of the field and will therefore diverge as $r_n \to 2m_n$.

The resulting scaling laws we have derived apply when the system is in static equilibrium just outside its Schwarzschild radius. By extension, they also apply at the point right before the system forms a black hole, if the collapse is quasistatic (i.e. the system is at equilibrium at all points during collapse). Any system which forms a black hole through a collapse off of equilibrium, will have an entropy bounded from above by equation (27), since the equilibrium configuration maximizes the entropy. Furthermore, for a system which forms a black hole, the generalized second law [1] implies that $\gamma A_n/(16\pi)$ is bounded from above by $A_n/4$, since otherwise it would not be entropically favorable for the system to form a black hole.

We have seen that the usual volume scaling properties of a system no longer hold when gravitational effects are taken into account. In the limit that our system approaches its Schwarzschild radius, its entropy scales as the area of the system. The temperature is also no longer an intensive variable, and scales inversely with the system's total energy. In other words, the system's thermodynamical quantities scale in the same manner as those of a black hole. These rather surprising results suggestions that at least to an extent, some of the intriguing aspects of black hole entropy can also be found in self-gravitating systems which do not possess a horizon. Once long range interactions are taken into account, the entropy of a system can behave in unfamiliar ways, and lead to scaling behavior which is no longer extensive.

The ability of high-energy theories such as loop quantum gravity and string theory to make predictions about the black hole entropy has generated much interest, and provides crucial insights into the entropy of black holes. The results in this work suggest that a wide class of systems will also result in the correct entropy scaling laws for black holes. It is hoped that by studying the entropy of these self-gravitating systems, we can learn more about black hole entropy. It is also hoped that studying the thermodynamics of self-gravitating systems will lead to a better understanding of the thermodynamics of more general non-extensive systems.

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